Perturbation and numerical analysis of suspended cables traveled by a single mass

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Keywords: Dynamics of suspended cables, moving masses, perturbation techniques.

SUMMARY. Oscillations induced by a moving mass on a moderately sagged suspended cable are studied. A linear continuum model is formulated, and both a standard Galerkin technique and a perturbation method are used to tackle the problem. This appear in the form of a parametric excitation problem, where the excitation frequency is related to the velocity of the travelling mass. A preliminary analysis is performed aimed to detect all the existing parametric excitation conditions of the mass-cable system, potentially leading to instability phenomena in which single modes are involved, or several modes interact.

1 INTRODUCTION

The dynamic of suspended cables has attracted the attention of many researchers as from the XVIII century (see, for instance, [1] for an historical reference). Cables are structural elements of great importance that find application in many engineering systems, among which suspended bridges, cableways and power transmission lines. Some studies are focused on the vibration analysis of suspended cables provided with fixed added masses, which can simulate sensors or non-structural elements (see, for instance, [2]-[3]). The addition of mass is a singularity source, that noticeably complicates the analysis of the system compared to the usual case of simple cable. In these last years the interest towards the dynamical response of structures excited by moving loads is widely increases, not only from a scientific point of view but also from a design perspective. The growth of this interest is to look into the design and construction of always more slender systems subjected to the action of travelling loads able to reach not negligible velocities. Many papers in the literature are devoted to the study of vibrations of elastic structures under moving loads (e.g., [4]); the behavior of taut strings has attracted attention as well, whereas to date dynamics of sagged suspended cables have been the subject of a remarkably lower number of papers. Wu & Chen [5] discuss dynamics of an horizontal extensible cable under a constant-velocity load; the hypothesized system is solved by using the finite element method comparing different mass models (only weight force and translational inertia, with the successive addition of centripetal and Coriolis accelerations). Wang [6] shows an analytical-numerical study of vibrations on a taut inclined cable due to a mass able to accelerate, that is modeled as a rigid body: the differential equations of the motion are deduced superimposing small displacements on the catenary state of the cable, by considering also the mass-cable friction; the spatial dependence is eliminated by the Galerkin method. Sofi & Muscolino [7] propose an improved series representation of vertical cable displacements in order to reproduce more correctly the abrupt changes in cable configuration due to the presence of the mass (singularity): in such a way they numerically analyze the in-plane response of suspended cables carrying an array of moving loads having arbitrary velocity, which are schematized as moving oscillator models.

In the present work, the small oscillations induced by a moving mass on a horizontal, sagged, suspended cable, are dealt with. The main goal is *not* the evaluation of the response, as usual in the literature, but the analysis of the critical conditions causing incipient instability of the system. The equations of motion of the cable bearing a single mass, moving with constant velocity, are deduced by a variational approach, under the usual hypothesis of small sag-to-span ratio. They are found to depend on four dimensionless parameters, two describing the cable characteristics (the Irvine parameter λ^2 and the damping function f_D), and two the mass features (the mass ratio μ and the velocity V). The analysis is carried out in two ways: (1) a numerical approach, based on a discrete Galerkin model and (2) an analytical approach, based on a perturbation method. The first analysis throws light on the mechanism leading to parametric excitation phenomena, and supplies a simple tool for detecting cable responses; the second analysis furnishes a parametric representation of the critical velocities and of the regions of instability.

2 THE CONTINUUM MODEL

Let us consider a cable as an elastic one-dimensional continuum of Cauchy, embedded in a two-dimensional space, subjected to its own weight, to inertial and viscous forces, traveled by a single point mass M. Two configurations of the cable are considered (Fig. 1): the reference (prestressed) configuration $\mathcal{C}_0$, occupied by the cable under its own load, and the actual configuration $\mathcal{C}$, occupied by the cable during the motion of the mass. Only oscillations of small amplitude are studied here, so that the two configurations are assumed adjacent. Let $\vec{u}(s_0,t) = u(s_0,t)\vec{a}_{t_0}(s_0) + v(s_0,t)\vec{a}_{n_0}(s_0)$ be the (small) displacement field leading $\mathcal{C}_0$ to $\mathcal{C}$, where $s_0 \in [0, L_0]$ is a curvilinear abscissa referred to the $\mathcal{C}_0$ configuration, t is the time and $\{\vec{a}_{t_0}(s_0), \vec{a}_{n_0}(s_0)\}$ the intrinsic basis.

The equations of motion of the cable-mass system are obtained by the extended Hamilton's principle:

$$\delta H = \int_{t_1}^{t_2} (\delta T - \delta U + \delta W)\, dt = 0 \tag{1}$$

where δT and δU are the first variations of the total kinetic energy T and of the (increment of the) total potential energy U, respectively, while δW is the work spent by the non-conservative damping forces f_D. By truncating the strain-displacement relationships at the second-order, they read:

$$U = \frac{1}{2}\int_0^{L_0} EA\left(u' - k_0 v\right)^2 ds_0 + \frac{1}{2}\int_0^{L_0} N_0\left(v' + k_0 u\right)^2 ds_0 + Mg\, v(\bar{s}_0(t), t) \tag{2}$$

$$T = \frac{1}{2}\int_0^{L_0} m\left(\dot{u}^2 + \dot{v}^2\right) ds_0 + \frac{1}{2}M\left[\frac{\mathrm{d}\,u(\bar{s}_0(t),t)}{\mathrm{d}t} + V\right]^2 + \frac{1}{2}M\left[\frac{\mathrm{d}\,v(\bar{s}_0(t),t)}{\mathrm{d}t}\right]^2 \tag{3}$$

$$\delta W = \int_0^{L_0} \left[f_{Du}(\dot{u},\dot{v})\,\delta\dot{u} + f_{Dv}(\dot{u},\dot{v})\,\delta\dot{v}\right] ds_0 \tag{4}$$

In Eqs (2) to (4): N_0 is the axial prestress of the cable, m the cable mass per-unit-length, EA the axial stiffness, $k_0(s_0)$ the cable curvature in $\mathcal{C}_0$ and g the gravity constant; $\bar{s}_0 = \bar{s}_0(t)$ is the instantaneous position of the mass M and $\mathrm{d}/\mathrm{d}t = V\partial/\partial s_0 + \partial/\partial t$ represents the total derivative, with $V := \mathrm{d}\bar{s}_0/\mathrm{d}t$ the instantaneous (relative) velocity of the mass M; the dots and dashes denote partial differentiation with respect to the time t and the curvilinear abscissa s_0, respectively; f_{Du} and f_{Dv} are mechanical damping functions, which depend on velocities.

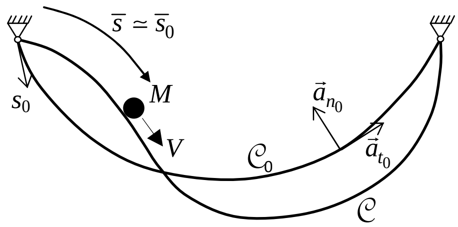

Figure 1: Sagged suspended cable subjected to a moving mass.

The following, simplifying assumptions, are introduced. (a) The two supports of the cable are at the same level (horizontal cable); (b) the sag-to-length ratio of the cable is small (i.e. $d_0/L_0 < 1/8$, d_0 being the cable sag), so that its static profile is approximated by a parabola with constant curvature $k_0 = 8d_0/L_0^2$; (c) the velocity $\vec{V}$ of the travelling mass M is assumed constant, therefore $\bar{s}_0 = Vt$; (d) the tangent component u of the displacement is statically condensed. The resulting (integro-differential) equation of motion is found to be:

$$m\ddot{v} - N_0 v'' + \frac{EAk_0^2}{L_0}\int_0^{L_0} v\,\mathrm{d}s_0 + f_{Dv}(\dot{v}) = 0 \quad \text{in } \mathfrak{D} \tag{5}$$

where $\mathfrak{D} = [0, \bar{s}_0^-] \cup [\bar{s}_0^+, L_0]$. The relevant mechanical and geometric boundary conditions are:

$$\begin{gathered} N_0\left[v'(\bar{s}_0^+, t) - v'(\bar{s}_0^-, t)\right] = Mg + M\left\{v''(\bar{s}_0, t)\,V^2 + 2V\,\dot{v}'(\bar{s}_0, t) + \ddot{v}(\bar{s}_0, t)\right\} \\ v(0,t) = v(L_0, t) = 0, \quad v(\bar{s}_0^+, t) - v(\bar{s}_0^-, t) = 0 \end{gathered} \tag{6}$$

to be enforced on $\partial\mathfrak{D} = 0 \cup \bar{s}_0 \cup L_0$. In particular, the mechanical condition (6_1) establishes the equilibrium at the (moving) singular point $\bar{s}_0$, at which the tangent $v'(\bar{s}_0, t)$ is discontinuous. It involves the gravitational term Mg, and the sum of three accelerations acting on the point mass M, namely, the centripetal, the Coriolis, and the driving acceleration due to cable motion, respectively. These latter terms can be indifferently evaluated, all together, at $\bar{s}_0^+$ or $\bar{s}_0^-$.

It is convenient to recast Eqs (5) and (6) in nondimensional form. The following positions are introduced:

$$\tilde{t} = \frac{1}{L_0}\sqrt{\frac{N_0}{m}}\,t, \quad \tilde{v} = \frac{v}{8d_0}, \quad \tilde{s}_0 = \frac{s_0}{L_0} \tag{7}$$

Omitting the tilde the equations of motion become:

$$
\begin{gathered}
\ddot{v} - v'' + \lambda^2 \int_0^1 v \,\mathrm{d}\, s_0 + f_{Dv}(\dot{v}) = 0 \quad \text{in } \mathfrak{D} \\
\left[v'(\bar{s}_0^{\,+}, t) - v'(\bar{s}_0^{\,-}, t) \right] = \mu^* + \mu \left\{ v''(\bar{s}_0, t)\, V^2 + 2V\, \dot{v}'(\bar{s}_0, t) + \ddot{v}(\bar{s}_0, t) \right\} \\
v(0,t) = v(1,t) = 0, \quad v(\bar{s}_0^{\,+}, t) - v(\bar{s}_0^{\,-}, t) = 0
\end{gathered}
\tag{8}
$$

where $\lambda^2 = EA/N_0 \cdot (8d_0/L_0)^2$ is the Irvine's parameter [1] governing the classical cable linear dynamics, $\mu = M/(mL_0)$ is the mass ratio, $\tilde{V} = V/\sqrt{N_0 / m}$ is the dimensionless velocity of mass, equal to the ratio between the true velocity and the celerity of the taut string. Moreover, the star superscript on μ is used for distinguishing gravitational from inertial masses; thus, the travelling force model is deduced from Eq. (8) by letting $\mu = 0, \mu^* \neq 0$.

3 DISCRETE MODEL

A discrete model is derived via the Galerkin approach. The resulting equations permits to highlight the role of the travelling mass on the system dynamic, and to numerically attack the problem. Since the goal of this paper is to analyze *global* behaviors of the system, the standard Galerkin method is applied, by renouncing to a more refined description of the *local* discontinuities, as e.g. performed in Ref [7]. The transversal displacement is assumed as:

$$
v(s_0, t) = \sum_{k=1}^{N} \phi_k(s_0)\, q_k(t) \tag{9}
$$

where ϕ_k are the eigenfunctions of the undamped cable (without traveling mass), taken from the linear theory (e.g., [1]) and q_k their time-varying amplitudes. The first N modes are taken into account (symmetric or anti-symmetric, uncoupled in the linear theory, or both). When the method of weighted residuals is applied to Eqs (8) (with damping neglected), one obtains:

$$
\int_{\mathfrak{D}} R_d\ \phi_j(s_0)\,\mathrm{d}\, D + \int_{\partial\mathfrak{D}} R_b\ \phi_j(s_0)\,\mathrm{d}\, S = 0 \tag{10}
$$

where R_d and R_b are the residuals in the domain and at the boundary. Using the orthogonality properties of eigenfunctions and reintroducing a modal damping in the discrete equations, the following N degree-of-freedom linear system is deduced:

$$
[\mathbf{M} + \mathbf{P}(t)]\ddot{\mathbf{q}}(t) + [\mathbf{C} + \mathbf{Q}(t)]\dot{\mathbf{q}}(t) + [\mathbf{K} + \mathbf{R}(t)]\mathbf{q}(t) = \mathbf{f}(t) \tag{11}
$$

where:

$$\mathbf{M}=\mathbf{I},\ \ \mathbf{C}=\mathrm{Diag}\left[2\xi_1\omega_1,...,2\xi_N\omega_N\right],\ \ \mathbf{K}=\mathrm{Diag}\left[\omega_1^2,...,\omega_N^2\right],\ \ f_j(t)=-\frac{\mu^*}{m_j}\phi_j(Vt)$$
$$P_{jk}(t)=\frac{\mu}{m_j}\phi_k(Vt)\phi_j(Vt),\ \ Q_{jk}(t)=\frac{2\mu}{m_j}V\phi_k'(Vt)\phi_j(Vt),\ \ R_{jk}(t)=\frac{\mu}{m_j}V^2\phi_k''(Vt)\phi_j(Vt) \tag{12}$$

$m_j=\int_0^1\phi_j^2\,\mathrm{d}s_0$ being the modal mass, ω_j the j-th circular frequency and ξ_j the j-th modal damping ratio. Equations (11)-(12) display the mechanism of the parametric excitation. Since the coefficients of the matrices **P**,**Q** and **R** are products of eigenfunctions (and their derivatives) sampled at a moving abscissa Vt, and since the eigenfunctions are linear combinations of harmonic functions (having spatial frequency ω_j or zero [1]), the matrices **P**,**Q** and **R** are multi-periodic functions of time, of (forcing) frequencies $\Omega_{jk}^{\pm}:=V\left|\omega_j\pm\omega_k\right|$ (for antisymmetric and symmetric modes) and $\Omega_j:=V\omega_j$ (for symmetric modes only). Therefore, parametric resonances involving one or more modes (combination resonances) can be predicted for critical values of the velocity V. Their study will be carried out ahead, via a perturbation approach.

Parametric excitation is exclusively due to inertial effects (indeed, it is proportional to μ). In particular, the centripetal acceleration modifies the structural stiffness, the Coriolis acceleration interacts with the mechanical damping and the driving acceleration changes the mass matrix. The classical term accounting for gravitation effect only appears as an externally applied force in the right member of these equations.

4 PERTURBATION SOLUTION

The Multiple Scale perturbation Method (MSM) is directly applied to the partial integro-differential equations (8), to find a first-order uniform expansion. The same results, of course, could be obtained by applying the MSM to the discretized equations (11). However, in view of further developments, the continuous approach is preferred here, since it is able to determine changes in spatial oscillation shapes at higher orders, which cannot be accounted for in the standard Galerkin approach.

4.1 *Formulation*

The displacement is expanded in series of a perturbation parameter ε<<1, $v(s_0,t)=v_0(s_0,t_0,t_1,t_2,...)+\varepsilon\,v_1(s_0,t_0,t_1,t_2,...)+...$, where independent time scales are introduced $t_h=\varepsilon^h t$ (h=0,1,2,...), so that the first and second time-derivatives are expressed as $\mathrm{D}=\mathrm{d}_0+\varepsilon\mathrm{d}_1+\varepsilon^2\mathrm{d}_2+\ldots$ and $\mathrm{D}^2=\mathrm{d}_0^2+2\varepsilon\mathrm{d}_0\mathrm{d}_1+\varepsilon^2(\mathrm{d}_1^2+2\varepsilon\mathrm{d}_0\mathrm{d}_2)+\ldots$, with $\mathrm{d}_h=\partial/\partial t_h$. Moreover, it is assumed that the (gravitational and inertial) mass ratios are small, $\mu=\varepsilon\hat{\mu},\mu^*=\varepsilon\hat{\mu}^*$, and the damping function is considered small of the same order ε, $f_D(\dot{v})=\varepsilon\hat{f}_D(\dot{v})$. The following perturbation equations are thus obtained:

order ε^0:

$$\mathrm{d}_0^2 v_0-v_0''+\lambda^2\int_0^1 v_0\,\mathrm{d}s_0=0\quad\text{in }\mathfrak{D}$$
$$v_0'(\bar{s}_0^+,t_0)-v_0'(\bar{s}_0^-,t_0)=0 \tag{13}$$
$$v_1(0,t_0)=v_1(1,t_0)=0,\quad v_0(\bar{s}_0^+,t)-v_0(\bar{s}_0^-,t)=0$$

order ε:

$$\mathrm{d}_0^2\, v_1 - v_1'' + \lambda^2 \int_0^1 v_1 \,\mathrm{d}\, s_0 = -2\,\mathrm{d}_0\,\mathrm{d}_1\, v_0 - \hat{f}_D(\mathrm{d}_0\, v_0) \quad \text{in } \mathfrak{D}$$

$$\left[v_1'(\bar{s}_0^{\,+}, t_0) - v_1'(\bar{s}_0^{\,-}, t_0)\right] = \hat{\mu}^* + \hat{\mu}\left[v_0''(\bar{s}_0, t_0)\, V^2 + 2V\,\mathrm{d}_0\, v_0'(\bar{s}_0, t_0) + \mathrm{d}_0^2\, v_0(\bar{s}_0, t_0)\right] \tag{14}$$

$$v_1(0, t_0) = v_1(1, t_0) = 0, \quad v_1(\bar{s}_0^{\,+}, t) - v_1(\bar{s}_0^{\,-}, t) = 0$$

Equations (13) admit the following generating solution:

$$v_0(s_0, t_0, t_1, ...) = \sum_{j=1}^{N} A_j(t_1, t_2)\, \phi_j(s_0)\, \mathrm{e}^{\mathrm{i}\,\omega_j\, t_0} + \text{c.c.} \tag{15}$$

where A_j are complex amplitudes, which are unknown functions of the slow times, and c.c. denotes the complex conjugate. By substituting Eq. (15) into the right side of Eqs (14), the projection of these forcing terms acting over $\mathfrak{D}$ and $\partial\mathfrak{D}$ on the k-th mode $\phi_k(s_0)$ leads to the following expression of the k-th generalized force:

$$\begin{aligned}
p_k(t_0, t_1, ...) = {} & -2\,\mathrm{i}\,\xi_k \omega_k^2 A_k\, \mathrm{e}^{\mathrm{i}\,t_0\omega_k} - 2\,\mathrm{i}\,\omega_k\, \mathrm{d}_1\, A_k\, \mathrm{e}^{\mathrm{i}\,t_0\omega_k} - \frac{\hat{\mu}^*}{2m_k}\left[\alpha_{k,0} + \left(\alpha_{k,11} + \mathrm{i}\,\alpha_{k,12}\right)\mathrm{e}^{\mathrm{i}\,t_0 V\omega_k}\right] + \\
& - \frac{\hat{\mu}}{4m_k}\sum_{j=1}^{N}\omega_j^2 A_j\left[\left(\alpha_{kj,21} + \mathrm{i}\,\alpha_{kj,22}\right)(V-1)^2\, \mathrm{e}^{\mathrm{i}\,t_0\left[\omega_j - V(\omega_j + \omega_k)\right]} + \right. \\
& + \left(\alpha_{kj,31} + \mathrm{i}\,\alpha_{kj,32}\right)(V+1)^2\, \mathrm{e}^{\mathrm{i}\,t_0\left[\omega_j + V(\omega_j - \omega_k)\right]} + \left(\alpha_{kj,31} - \mathrm{i}\,\alpha_{kj,32}\right)(V-1)^2\, \mathrm{e}^{\mathrm{i}\,t_0\left[\omega_j - V(\omega_j - \omega_k)\right]} + \\
& \left. + \left(\alpha_{kj,21} - \mathrm{i}\,\alpha_{kj,22}\right)(V+1)^2\, \mathrm{e}^{\mathrm{i}\,t_0\left[\omega_j + V(\omega_j + \omega_k)\right]}\right] + \\
& - \frac{\hat{\mu}}{2m_k}\sum_{j=1}^{N}\omega_j^2 A_j\left[\left(\alpha_{kj,41} + \mathrm{i}\,\alpha_{kj,42}\right)\mathrm{e}^{\mathrm{i}\,t_0\left[\omega_j - V\omega_k\right]} + \left(\alpha_{kj,41} - \mathrm{i}\,\alpha_{kj,42}\right)\mathrm{e}^{\mathrm{i}\,t_0\left[\omega_j + V\omega_k\right]}\right] + \\
& - \frac{\hat{\mu}}{2m_k}\sum_{j=1}^{N}\omega_j^2 A_j\left[\left(\alpha_{kj,41} + \mathrm{i}\,\alpha_{kj,52}\right)(V-1)^2\, \mathrm{e}^{\mathrm{i}\,t_0\left[\omega_j - V\omega_j\right]} + \right. \\
& \left. + \left(\alpha_{j,41} - \mathrm{i}\,\alpha_{kj,52}\right)(V+1)^2\, \mathrm{e}^{\mathrm{i}\,t_0\left[\omega_j + V\omega_j\right]}\right] + \frac{\hat{\mu}}{m_k}\sum_{j=1}^{N}\alpha_{kj,41}\omega_j^2 A_j\, \mathrm{e}^{\mathrm{i}\,t_0\omega_j} + \text{c.c.}
\end{aligned} \tag{16}$$

where the damping function f_D associated to ϕ_j has been considered orthogonal to ϕ_k. The coefficients α in Eq. (16) are related to the cable modal shapes (their expressions are here not reported for the sake of brevity). The general form (16) is derived from symmetric modes; when anti-symmetric modes are considered, the sole first summation remains (according to the fact that anti-symmetric modes have zero mean value). This circumstance leads to some differences in the evaluation of resonant terms.

4.2 *Critical velocity analysis*

In order to eliminate secular terms in Eq. (14), it needs to remove the harmonic ω_k in Eq. (16). The first summation (and related c.c.) is examined first. By defining $\Omega_{jk}^{\pm} := V\left|\omega_j \pm \omega_k\right|$ as forcing frequencies, it appears that *summed combination resonance* ($\omega_j + \omega_k = \Omega_{jk}^{\pm}$) or *difference combination resonance* ($\left|\omega_j - \omega_k\right| = \Omega_{jk}^{\pm}$) occur [8,9], when V assumes one of the following critical values:

$$V_{cr} = \frac{\left|\omega_j \pm \omega_k\right|}{\left|\omega_j \pm \omega_k\right|} \tag{17}$$

Equation (17) defines *four* critical velocities valid for both symmetric and anti-symmetric modes. Modal interactions (i.e. combination resonance) can occur for any V_{cr}; single-mode parametric excitation can in principle occur just when V_{cr}=1, but relating terms in Eq. (16) actually disappear owing to the balancing of the three inertial terms. Of course, V_{cr}=1 also produces external resonance, via the gravitational mass; this latter, however, is responsible for response magnification only.

The second and third summation (and related c.c.) in Eq. (16) are now checked. If the forcing frequency is defined as $\Omega_j := V\omega_j$, summed or difference combination resonances ($\left|\omega_j \pm \omega_k\right| = \Omega_j$) occur at the critical velocities:

$$V_{cr} = \frac{\left|\omega_j \pm \omega_k\right|}{\omega_j} \tag{18}$$

Therefore, *four* additional critical velocities ($\left|\omega_j \pm \omega_k\right| / \omega_j, \left|\omega_j \pm \omega_k\right| / \omega_k$) exist for symmetric motions only, *all different from 1*, involving the *j*-th and *k*-th modes, except for 1:2 internal resonance conditions ($\omega_k=2\omega_j$) that imply V_{cr}=1: in this specific case a difference combination resonance appears in place of the single-mode excitation. Modal interaction manifests for any V_{cr}, whereas parametric resonance on a single mode can just occur when V_{cr}=2.

It is important to note that, for all critical velocities, Eqs (17) and (18), $V_{cr} < 1$ entails difference combination resonance, and $V_{cr} > 1$ summed combination resonance.

The antisymmetric natural frequencies, equal to $\omega_j=2\pi j$, j=1,2,... (coincident with that of the taut string), are independent of the cable elasto-geometric quantities, and therefore also the critical velocities posses this property. In contrast, the symmetric frequencies do depend on the λ^2 Irvine's parameter, and they are different from that of the taut string, for which $\omega_j=(2j-1)\pi$. By taking, e.g., λ^2=20 (i.e. below the first cross-over point), one obtains ω_j=5.06, 9.54, 15.73, 22.00, $\sim 9\pi$, $\sim 11\pi$, $\sim 13\pi$, $\sim 15\pi$. Table 1 shows the critical velocities relevant to the first eight anti-symmetric modes; Table 2 collects the critical velocities for the first eight symmetric modes when λ^2=20. The more significant results is the existence of a large number of parametric internal resonances as a function of the velocity of the travelling mass, similarly to the case of the elastic beam subjected to moving masses [10].

5 INSTABILITY REGION ANALYSIS

Once the (V-dependent) resonant terms in Eqs (16) are zeroed, *Amplitude Modulation Equations* (AME) governing the motion of the complex amplitudes on the t_1-scale are obtained. When the perturbation parameter is reabsorbed, and coming back to the true time t, they appear in the following form:

$$\dot{\mathbf{A}} = \mathbf{L}_1\mathbf{A} + \mathbf{L}_2\bar{\mathbf{A}} + \mathbf{f} \tag{19}$$

where **A** is a vector collecting the amplitudes, and $\mathbf{L}_1$, $\mathbf{L}_2$ are complex matrices depending on the cable and mass parameters. In particular, in order to investigate the neighborhood of the critical velocities, a (small) detuning parameter $\sigma := (V - V_{cr})/V_{cr}$ is introduced. Finally, **f** is a vector of known terms, dependent on the gravitational mass ratio μ^* and σ; it is different from zero

V_{cr}	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8
***j*=1**	1							
***j*=2**	$1/3$,1,3	1						
***j*=3**	$1/2$,1,2	$1/5$,1,5	1			Sym		
***j*=4**	$3/5$,1,$5/3$	$1/3$,1,3	$1/7$,1,7	1				
***j*=5**	$2/3$,1,$3/2$	$3/7$,1,$7/3$	$1/4$,1,4	$1/9$,1,9	1			
***j*=6**	$5/7$,1,$7/5$	$1/2$,1,2	$1/3$,1,3	$1/5$,1,5	$1/11$,1,11	1		
***j*=7**	$3/4$,1,$4/3$	$5/9$,1,$9/5$	$2/5$,1,$5/2$	$3/11$,1,$11/3$	$1/6$,1,6	$1/13$,1,13	1	
***j*=8**	$7/9$,1,$9/7$	$3/5$,1,$5/3$	$5/11$,1,$11/5$	$1/3$,1,3	$3/13$,1,$13/3$	$1/7$,1,7	$1/15$,1,15	1

Table 1: Critical velocities for the first eight anti-symmetric modes of the cable.

V_{cr}	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8
***j*=1**	1,2							
***j*=2**	0.31,0.47, 0.89,1,1.53 2.89,3.25	1,2						
***j*=3**	0.51,0.68,1 1.32,1.95 2.11,4.11	0.24,0.39, 0.65,1,1.61, 2.65,4.08	1,2			Sym		
***j*=4**	0.63,0.77,1 1.23,1.60 3.35,5.35	0.39,0.57,1 1.30,1.43 2.53,3.30	0.17,0.28 0.40,1,1.71 2.40,6.02	1,2				
***j*=5**	0.70,0.82,1 1.18,1.44 4.59,6.59	0.49,0.66 1,1.34,1.96 2.02,3.96	0.28,0.44 0.80,1,1.56 2.80,3.51	0.12,0.22, 0.28,1,1.78 2.28,8.01	1,2			
***j*=6**	0.74,0.85,1 1.15,1.34 5.83,7.83	0.57,0.72 1,1.28,1.76 2.62,4.62	0.37,0.54 1,1.20,1.45 2.67,3.20	0.22,0.36 0.57,1,1.64 2.57,4.50	$1/10$, $2/11$ $2/9$,1, $20/11$ $20/9$,10	1,2		
***j*=7**	0.78,0.88,1 1.12,1.28 7.08,9.08	0.62,0.77, 1,1.23,1.61 3.28,5.28	0.44,0.61,1 1.38,1.60, 2.25,3.60	0.30,0.46 0.86,1,1.54 2.86,3.33	$2/11$, $4/13$ $4/9$,1, $22/13$ $22/9$, $11/2$	$1/12$, $2/13$ $2/11$,1, $24/13$, $24/11$,12	1,2	
***j*=8**	0.81,0.89,1 1.11,1.24 8.32,10.32	0.66,0.80, 1,1.20,1.51 3.94,5.94	0.50,0.67 1,1.33,1.996, 2.002,4.00	0.36,0.53,1 1.14,1.47 2.75,3.14	$1/4$, $2/5$ $2/3$,1 $8/5$, $8/3$,4	$2/13$, $4/15$ $4/11$,1, $26/15$, $26/11$, $13/2$	$1/14$, $2/15$ $2/13$,1, $28/15$ $28/13$,14	1,2

Table 2: Critical velocities for the first eight symmetric modes of the cable (λ^2=20).

only when $V_{cr} \simeq 1$.

Inspection of the coefficients of the $\mathbf{L}_1$, $\mathbf{L}_2$ matrices (not reported here) reveals that: (a) if the velocity is *non-critical*, the AME contain damping terms only so that all the amplitudes decay in time (this denoting that the response is of ε-order, i.e. of the same order of the excitation); (b) if, in contrast, the velocity is equal to a critical value, coupling terms appear in the AME, potentially leading to order-1 responses (i.e. much larger than the excitation). Anyway, at particular critical velocity, just few modes participate to the leading motion, according to Tables 1-2, so that the analysis can be limited to these interacting components, since the remaining ones decay.

Since the AME are a linear homogeneous problem (when $V_{cr} \neq 1$), the response is governed by its eigenvalues only, so that the motion either extinguishes (stable equilibrium) or diverges (unstable equilibrium) in time. By evaluating the couples of values (μ,σ) for which the real part of the eigenvalues vanish, the boundaries of the instability regions are determined. Of course, the linear analysis performed here just permits to check the stability, not to evaluate the (post-critical) response in the unstable region, for which a nonlinear analysis is necessary.

A further analysis of the coefficients shows that: if $V_{cr} < 1$, then $\mathbf{L}_2=\mathbf{0}$, whereas if $V_{cr} > 1$ then $\mathbf{L}_2 \neq 0$, so that the AME appear as generalization of certain coupled Hill's equations discussed in the literature [9]. According to those results, close to difference combination resonances ($V_{cr} < 1$) the response is stable, while close to summed combination resonances ($V_{cr} > 1$) instability domains do exist. A deeper investigation, however, is necessary to confirm this property.

As a first example, a summed combination resonance is analyzed concerning the antisymmetric modes. Setting the velocity at the critical value $V_{cr} = 2$, from Table 1 two cases are pointed out: resonance between modes (1,3) and resonance between modes (2,6). Obviously, other resonances would appear if Table 1 were extended (e.g. (3,9)), with the further consequence that resonances involving three or more modes (e.g.,(1,3,9)) would occur. Here, however, the truncation at the first eighth modes is imposed. Damping ratio is selected according to the Rayleigh model, setting it equal to 0.5% for the first two modes. Figure 2.a shows the instability domain for the modes 1,3 (continuous line) and for the modes 2,6 (dashed lines); bold round points concern numerical solutions, obtained by a direct integration of the Galerkin discrete model (11)-(12), reduced to modes 1,3. Figures 2.b and 2.c presents two time-histories deriving from the integration of the discrete model for μ=0.08, V=V_A=1.56 and V=V_B=1.7, respectively, using an integration time equal to 50 times the traveling time T=1/V of the mass (in practice, the load is considered periodically appearing again on the cable at the end of each travel). The agreement between numerical and perturbation results appears remarkable. The inclination of the instability region is due to forcing zero-frequencies $\Omega_{kk}^{-} := V\left|\omega_k - \omega_k\right| = 0, \forall V$ in Eq. (16). The relevant terms account for the increment of modal mass (see P_{kk} in Eq. (12_5)) and for the reduction of the modal stiffness (see R_{kk} in Eq. (12_7)) produced by M, that both reduce the natural frequencies and, consequently, the critical velocities.

6 CONCLUSIONS

In this paper a linear continuum model for a travelling mass on a suspended horizontal cable is proposed. The first results obtained through a perturbation technique highlight as the traveling mass model shows not only the classical external resonance (due to the gravitational load) but also a large number of parametric internal resonances between cable modes, which allows to identify several critical velocities leading to possible dynamic instability. In fact, the effective loss of stability may be insignificant because of the finite length of the structure and the consequent limited time duration of excitation; nevertheless, the response amplification appears of vital interest both from a conceptual point of view and, also in practice, for series of moving loads traversing the cable. A preliminary analysis, devoted to a specific instability domain, points out the good accuracy of the perturbation solution compared to direct numerical integrations of the discrete equations of motion.

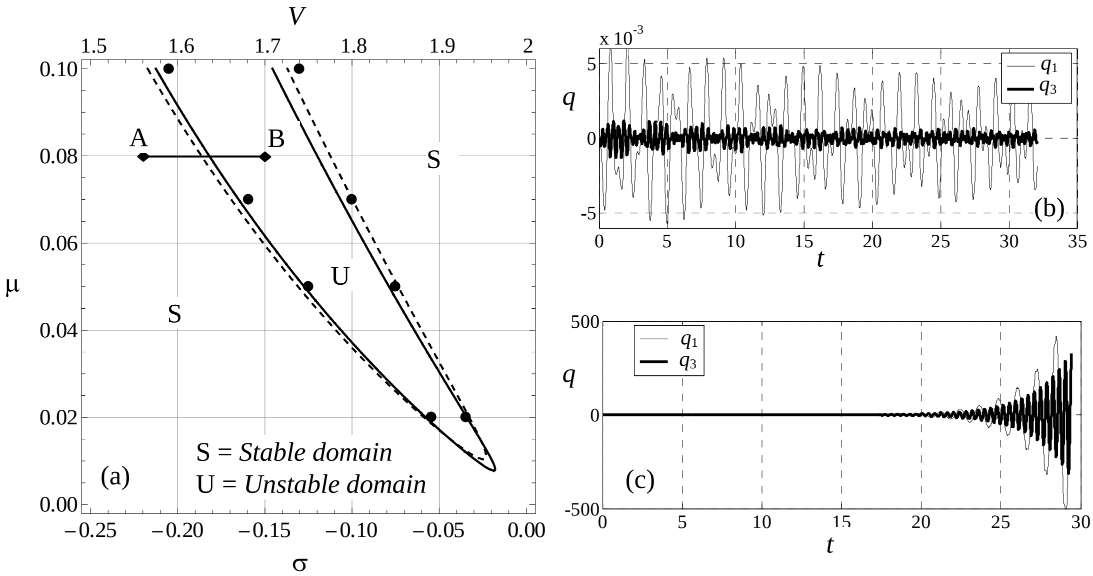

Figure 2: (a) Instability regions ($V_{cr} = 2$), (b)-(c) Time-histories related to A- and B-values.

Acknowledgements

This work has been partially supported by the Italian Ministry of University and Research (MIUR) through a national PRIN 2007 co-financed program, and by the University of Genoa. The authors wish to give particular thanks to Mr. Andrea Di Mattia who found preliminary results on this subject during his graduation thesis.

References

[1] Irvine, H.M., *Cable Structures*, MIT Press, Cambridge (1981).

[2] Lin, H.P., Perkins, N.C., "Free vibration of complex cable/mass systems: theory and experiment," *Journal of Sound and Vibration*, **179**(1), 131-149 (1995).

[3] Al-Qassab, M., Nair, S., "Wavelet-Galerkin method for the free vibrations of an elastic cable carrying an attached mass," *Journal of Sound and Vibration*, **270**(1-2), 191-206 (2004).

[4] Frýba, L., *Vibrations of Solids and Structures under Moving Loads*, Thomas Telford, London (1999).

[5] Wu, J.S., Chen, C.C., "The dynamic analysis of a suspended cable due to a moving load," *International Journal for Numerical Methods in Engineering*, **28**(10), 2361-2381 (1989).

[6] Wang, Y.M., "The transient dynamics of a cable-mass system due to the motion of an attached accelerating mass," *International Journal of Solids and Structures*, **37**(9), 1361-1383 (2000).

[7] Sofi, A., Muscolino, G., "Dynamic analysis of suspended cables carrying moving oscillators," *International Journal of Solids and Structures*, **44**(21), 6725-6743 (2007).

[8] Seyranian, A.P., Mailybaev, A.A., *Multiparameter Stability Theory with Mechanical Applications*, Series A, Vol. 13, World Scientific, Singapore (2003).

[9] Nayfeh, A.H., Mook, D.T., *Nonlinear Oscillations*, John Wiley & Sons, New York (1979).

[10] Rao, G.V., "Linear dynamics of an elastic beam under moving loads," *Journal of Vibration and Acoustics*, ASME, **122**(3), 281-289 (2000).